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Solid-body-type vortex solutions of the Euler equations

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A class of unsteady vortex solutions of the Euler equations is investigated. The solutions satisfy the von Kármán–Bödewadt similarity scalings and correspond to free and forced oscillations of radially unbounded solid-body-type vortices with axially varying rotation rates. The vortices may be of unbounded vertical extent or confined by impermeable top and/or bottom plates. In the latter case the bounding plates may be stationary or oscillatory. A solution breakdown result tends to support the hypothesis that breakdown of viscous Bödewadt-type counter-rotating vortex flows is an essentially inviscid process.

1. Introduction

The pure solid-body vortex is an exact solution of the steady-state Euler equations (and Navier–Stokes equations) characterized by a spatially uniform angular velocity. On the large scale, atmospheres and oceans as a whole tend to rotate with their planets, that is, as solid bodies. On the small scale, the core regions of separated vortices and turbulent vortices fall under control of the diffusion process and tend to rotate as solid bodies (Lugt 1983). A number of vortex solutions of the Navier–Stokes equations have inner cores in approximate solid-body rotation, for example, the decaying vortices of Lamb (1945) and Taylor (1918), the convective/diffusive vortices of Burgers (1948), Rott (1958, 1959), Sullivan (1959) and Bellamy-Knights (1970, 1971), and the conically similar viscous flows of Long (1958, 1961) and Yih *et al.* (1982).

If the strength of a solid-body vortex varies along its rotation axis, the associated pressure field also varies along the axis and a meridional circulation is induced. Such is the case with rotating flow over an infinite stationary disk (Bödewadt 1940), an infinite rotating disk in a fluid otherwise at rest (von Kármán 1921), and flows between infinite rotating coaxial disks (Batchelor 1951; Stewartson 1953). Axial variations in angular velocity in these von Kármán–Bödewadt-type flows are a consequence of no-slip boundary conditions. The induced circulations are such that the radial and azimuthal velocity components vary linearly with radius, while the vertical velocity is independent of radius. The same spatial dependences apply to the time-dependent versions of these flows (Pearson 1965; Bodonyi & Stewartson 1977; Bodonyi 1978). Steady and unsteady von Kármán–Bödewadt-type flows are reviewed by Zandbergen & Dijkstra (1987).

More recently, Shapiro & Markowski (1999) and Shapiro (2001) obtained exact unsteady solutions of the Euler equations for flows satisfying the von Kármán–Bödewadt velocity scalings. In these inviscid flows, axial variations of angular velocity are prescribed in the initial conditions (since they can no longer arise from the boundary conditions). Shapiro & Markowski (1999) considered vertically discontinuous vortices consisting of piecewise-constant layers of angular velocity. These ‘layered’ solutions are generally oscillatory, but a singular solution was found for a vertically unbounded vortex overlying a layer of non-rotating fluid, and a well-behaved monotonic solution was found for a vertically confined vortex overlying non-rotating fluid. A continuous solution for an axially propagating centrifugal wave was described by Shapiro (2001).

The present study extends and generalizes the analysis of Shapiro & Markowski (1999). In § 2 we introduce the von Kármán–Bödewadt velocity scalings and discuss their consequences for inviscid vortices. In § 3.1 we derive a general closed-form solution of the Euler equations for a vertically infinite ‘primary vortex’ overlying a ‘boundary layer vortex’ of finite depth and general (continuous or discontinuous) initial vertical distribution of angular velocity. Also in § 3.1, we obtain a solution breakdown result for counter-rotating flows. Examples of a singular solution and a well-behaved solution are presented in § 3.2 and § 3.3, respectively. Several aspects of the counter-rotating flow breakdown described in § 3.2 are in good agreement with the breakdown observed in viscous Bödewadt-type counter-rotating vortex flows (Bodonyi & Stewartson 1977; Bodonyi 1978). Forced oscillations of vertically confined vortices by the normal motion of an infinite horizontal top plate are examined in § 4. A simple solution for a single-layer vortex subjected to general plate forcing is presented in § 4.1, and a numerical solution for a two-layer vortex forced by a plate in simple harmonic motion is presented in § 4.2.

2. von Kármán–Bödewadt similarity principle

2.1. *Eulerian framework*

Our flows satisfy the axisymmetric Euler equations and the incompressibility condition,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = 0, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{3}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

Here u, v and w are the radial (r), azimuthal (ϕ) and vertical (z) velocity components, respectively, ρ is the (constant) density and p is the perturbation pressure (pressure minus hydrostatic pressure).

The initial state consists of a solid-body vortex with axially varying angular velocity, a radial velocity that varies linearly with radius and, in view of (4), a vertical velocity that is independent of radius. Inspection of (1)–(4) shows that these spatial dependences persist, that is, the flows satisfy the von Kármán–Bödewadt similarity relations,

$$u = r\,F(z,t), \quad v = r\,\Omega(z,t), \quad w = H(z,t), \tag{5}$$

with

$$F = -\frac{1}{2}\frac{\partial H}{\partial z}. \tag{6}$$

Accordingly, the vertical velocity, horizontal divergence $\delta\,(=\partial u/\partial r + u/r = -\partial H/\partial z)$ and vertical vorticity $\zeta\,(=\partial v/\partial r + v/r = 2\Omega)$ are all independent of radius.

Equations (5) and (6) reduce (1)–(3) to simpler forms,

$$\frac{r}{2}\left[-\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial z}\right)+\frac{1}{2}\left(\frac{\partial H}{\partial z}\right)^2 - H\frac{\partial^2 H}{\partial z^2} - 2\Omega^2\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{7}$$

$$\frac{\partial \Omega}{\partial t} = \Omega\frac{\partial H}{\partial z} - H\frac{\partial \Omega}{\partial z}, \tag{8}$$

$$\frac{\partial H}{\partial t} + H\frac{\partial H}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{9}$$

An azimuthal vorticity equation is obtained by cross-differentiating (7) and (9) to eliminate p,

$$\frac{\partial}{\partial z}\left[-\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial z}\right)+\frac{1}{2}\left(\frac{\partial H}{\partial z}\right)^2 - H\frac{\partial^2 H}{\partial z^2} - 2\Omega^2\right] = 0. \tag{10}$$

It will be convenient later to consider a rewritten form of (10) and an integrated form of (10):

$$\frac{\partial^3 H}{\partial t\,\partial z^2} + H\frac{\partial^3 H}{\partial z^3} + 4\Omega\frac{\partial \Omega}{\partial z} = 0, \tag{11}$$

$$-\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial z}\right)+\frac{1}{2}\left(\frac{\partial H}{\partial z}\right)^2 - H\frac{\partial^2 H}{\partial z^2} - 2\Omega^2 = -C(t), \tag{12}$$

where $C(t)$ is a function of integration. The pressure field is obtained from (7), (9) and (12) as

$$p = p_s - \rho\int_{z_b}^{z}\frac{\partial H}{\partial t}\,\mathrm{d}z' + \tfrac{1}{2}\rho(\dot{z}_b^2 - H^2) + \tfrac{1}{4}\rho\,C(t)\,r^2, \tag{13}$$

where z_b is a lower reference surface, for example a stationary plate or a plate moving with speed $\dot{z}_b$ (we use a dot to denote total differentiation on a boundary), and p_s is a stagnation pressure. In view of (13), the radial pressure gradient force is independent of height and equal to $-C(t)\,r/2$.

We seek an expression for $C(t)$ for the general case where both top (z_t) and bottom (z_b) plates may be present (and possibly in vertical motion). Integrating (12) from z_b to z_t, we obtain

$$C(t) = \frac{1}{z_t - z_b}\left[\ddot{z}_t - \ddot{z}_b + \int_{z_b}^{z_t}(2\Omega^2 - 3\delta^2/2)\,\mathrm{d}z\right]. \tag{14}$$

Thus our problem reduces to solving two coupled partial differential equations, (8) and (11), or, equivalently, (8) and the integro-differential equation arising from applying (14) in (12).

2.2. *Lagrangian framework*

In the Lagrangian framework, we regard $z = z(t, z_0)$ as a dependent variable, and t and the initial height $z_0 = z(0, z_0)$ as independent variables. Accordingly, we are concerned with the motion of horizontal material surfaces. The Lagrangian framework simplifies the analysis and leads to a closed-form solution for the special case where C is constant (see § 3).

Introducing the total derivative operator, $\mathrm{d}/\mathrm{d}t = \partial/\partial t + H\,\partial/\partial z$, we can write (12) and (8) as

$$\mathrm{d}\delta/\mathrm{d}t + \delta^2/2 - 2\Omega^2 = -C(t), \tag{15}$$

$$\mathrm{d}\Omega/\mathrm{d}t = -\Omega\delta. \tag{16}$$

Equations (15) and (16) are (19) and (18) in Shapiro & Markowski (1999) but now valid for a continuum, not just for discrete fluid layers of piecewise-constant angular velocity and divergence.

It will be convenient to work with a new dependent variable ϕ defined by

$$\phi = |\Omega|^{-1/2}, \tag{17}$$

or $|\Omega| = 1/\phi^2$. In view of (16), δ is related to ϕ by

$$\delta = (2/\phi)\,\mathrm{d}\phi/\mathrm{d}t. \tag{18}$$

It also follows from (16) that $\Omega(t, z_0) = \Omega(0, z_0)\exp(-\int_0^t \delta(t', z_0)\,\mathrm{d}t')$, and thus the sense of rotation of a material surface does not change. Applying (17) and (18) in (15) yields

$$\mathrm{d}^2\phi/\mathrm{d}t^2 - \phi^{-3} = -\tfrac{1}{2}C(t)\,\phi. \tag{19}$$

To obtain the vertical 'stretching' experienced by adjacent horizontal material surfaces, first integrate $\mathrm{d}z/\mathrm{d}t = H$ to obtain the height of a horizontal material surface $z = z(t, z_0)$,

$$z(t, z_0) = z_0 + \int_0^t H(t', z_0)\,\mathrm{d}t'. \tag{20}$$

Next, differentiate (20) with respect to z_0, and use $\partial H/\partial z_0 = (\partial H/\partial z)(\partial z/\partial z_0) = -\delta\,\partial z/\partial z_0$ to obtain

$$\frac{\partial z}{\partial z_0} = 1 - \int_0^t \delta(t', z_0)\frac{\partial z}{\partial z_0}\,\mathrm{d}t'. \tag{21}$$

Taking the total derivative of (21) yields

$$(\mathrm{d}/\mathrm{d}t)(\partial z/\partial z_0) = -\delta\,\partial z/\partial z_0. \tag{22}$$

Applying (18) in (22), integrating, and using the fact that $\partial z/\partial z_0(0, z_0) = 1$, we get

$$\partial z/\partial z_0 = \phi_0^2/\phi^2, \tag{23}$$

where $\phi_0 \equiv \phi(0, z_0)$ (more generally, a subscript 0 on any dependent variable denotes the initial value of that variable). Applying (17) in (23) (and using the fact that $|\Omega|/|\Omega_0| = \Omega/\Omega_0$) yields

$$\partial z/\partial z_0 = \Omega/\Omega_0. \tag{24}$$

The quantity $\partial z/\partial z_0$ is the spacing Δz between two horizontal material surfaces normalized by their initial spacing Δz_0 evaluated locally, that is, in the limit $\Delta z_0 \to 0$. Thus, a value of $\partial z/\partial z_0 > 1$ indicates that neighbouring horizontal material surfaces have been stretched from their initial configuration. According to (24), the ratio of vertical vorticity Ω to 'thickness' Δz is invariant (in the limit $\Delta z_0 \to 0$). We thus regard (24) as an equation for potential vorticity conservation.

Integrating (24) with respect to z_0 yields z as

$$z(t, z_0) = z_b(t) + \int_{z_b(0)}^{z_0} \frac{\Omega(t, z_0')}{\Omega_{0'}}\,\mathrm{d}z_0', \tag{25}$$

where $\Omega_{0'} \equiv \Omega(0, z_0')$ defines the subscript $0'$ dummy notation.

To obtain the vertical velocity, take the total derivative of (25) and apply (16),

$$H(t, z_0) = \dot{z}_b(t) - \int_{z_b(0)}^{z_0} \delta(t, z_0') \frac{\Omega(t, z_0')}{\Omega_{0'}} \, \mathrm{d}z_0'. \tag{26}$$

$C(t)$ can be expressed in Lagrangian form by rewriting (14) as

$$C(t) = \frac{\ddot{z}_t - \ddot{z}_b}{z_t - z_b} + \frac{1}{z_t - z_b} \int_{z_b(0)}^{z_t(0)} [2\Omega^2(t, z_0') - \tfrac{3}{2}\delta^2(t, z_0')] \frac{\Omega(t, z_0')}{\Omega_{0'}} \, \mathrm{d}z_0'. \tag{27}$$

A single integro-differential equation for ϕ results from applying (17), (18) and (27) in (19),

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} - \phi^{-3} = -\frac{\phi}{2}\left(\frac{\ddot{z}_t - \ddot{z}_b}{z_t - z_b}\right) - \frac{\phi}{z_t - z_b} \int_{z_b(0)}^{z_t(0)} [1 - 3(\phi \, \mathrm{d}\phi/\mathrm{d}t)^2] \frac{\phi_{0'}^2}{\phi^6} \, \mathrm{d}z_0'. \tag{28}$$

Finally, we derive a conservation principle for the (squared) meridional vorticity components. Writing (11) as $\frac{1}{4}(\mathrm{d}/\mathrm{d}t)(\partial\delta/\partial z) = \Omega \, \partial\Omega/\partial z$ and multiplying by $\partial\delta/\partial z$ yields

$$\frac{1}{4}\frac{\partial\delta}{\partial z}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\delta}{\partial z}\right) = \Omega\frac{\partial\delta}{\partial z}\frac{\partial\Omega}{\partial z}. \tag{29}$$

Multiplying the vertical derivative of (8) by $\partial\Omega/\partial z$ and rearranging yields

$$\frac{\partial\Omega}{\partial z}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Omega}{\partial z}\right) = -\Omega\frac{\partial\delta}{\partial z}\frac{\partial\Omega}{\partial z}. \tag{30}$$

Adding (29) to (30) yields $(\mathrm{d}/\mathrm{d}t)[\frac{1}{4}(\partial\delta/\partial z)^2 + (\partial\Omega/\partial z)^2] = 0$, which integrates to

$$\tfrac{1}{4}(\partial\delta/\partial z)^2 + (\partial\Omega/\partial z)^2 = B(z_0). \tag{31}$$

In terms of the meridional vorticity components (radial vorticity $\xi = -\partial v/\partial z = -r \, \partial\Omega/\partial z$ and azimuthal vorticity $\eta = \partial u/\partial z = r/2 \, \partial\delta/\partial z$), (31) becomes

$$\eta^2 + \xi^2 = r^2 B(z_0). \tag{32}$$

3. Closed-form solution for boundary-layer-type vortices

3.1. *General solution*

We now examine the special case of an upper 'primary vortex' overlying a lower 'boundary-layer vortex'. The lower vortex is considered to have a finite vertical thickness and an arbitrary continuous or discontinuous initial vertical profile of divergence and angular velocity. It is considered to be bounded from below by an impermeable plate that may or may not be stationary, and bounded from above by the upper vortex. The upper vortex is assumed to be (i) of infinite vertical extent ($z_t = \infty$), (ii) non-divergent, and (iii) in solid-body rotation with spatially constant angular velocity (say, $\bar{\Omega}$). Without loss of generality we assume that $\bar{\Omega} > 0$. It can readily be shown that these conditions persist for all time, and therefore C $(=2\bar{\Omega}^2)$ is also temporally constant. In view of (13) and the constancy of C, the radial pressure gradient force is independent of z and t. We use the designation 'boundary-layer vortex' for the lower vortex because of its relative shallowness (the ratio of lower vortex thickness to upper vortex thickness is zero) and the fact that the radial pressure gradient force in the upper flow is impressed upon the lower flow, as in conventional boundary layer theory (Schlichting 1979).

It should be noted that vortices confined by both top and bottom plates are generally associated with a temporally varying C, even when both plates are stationary. Perhaps the simplest example of this is a two-layer flow (vortex overlying non-rotating fluid) confined between stationary plates, described in § 4.d of Shapiro & Markowski (1999). Using the data provided in figure 3 of that paper we see that C is non-zero initially but approaches the steady-state value of zero as $t \to \infty$. It should also be noted that not all constant-C flows are of boundary-layer type. For example, the axially propagating centrifugal wave solution of Shapiro (2001) is associated with a constant C (twice the difference between the (squared) base state and disturbance angular velocities) but would probably not be regarded as a boundary-layer-type flow.

As we will now see, the condition that $C = 2\bar{\Omega}^2$ leads to a closed-form solution of (19) for arbitrary initial profiles of boundary layer vortex divergence and angular velocity. Changing the dependent variable to P ($\equiv \mathrm{d}\phi/\mathrm{d}t$) and regarding ϕ as a new independent variable, (19) becomes $\mathrm{d}P^2/\mathrm{d}\phi = 2\phi^{-3} - 2\bar{\Omega}^2\phi$. Integrating and taking the square root, we obtain

$$\mathrm{d}\phi/\mathrm{d}t = \pm\sqrt{-\phi^{-2} - \bar{\Omega}^2\phi^2 + 2\bar{\Omega}D_0}. \tag{33}$$

Here $2\bar{\Omega}\,D_0$ is a constant of integration, the factor $2\bar{\Omega}$ introduced for later convenience. D_0 can be related to the initial conditions by applying (17) and (18) in (33) at the initial time,

$$D_0 = \frac{|\Omega_0|}{2\bar{\Omega}} + \frac{\bar{\Omega}}{2|\bar{\Omega}_0|} + \frac{\delta_0^2}{8\bar{\Omega}|\Omega_0|}. \tag{34}$$

Separating variables in (33), integrating and rearranging the result yields

$$\phi^2 = \frac{1}{\bar{\Omega}}\left[D_0 + \sqrt{D_0^2 - 1}\,\sin(2\bar{\Omega}t + A_0)\right], \tag{35}$$

where the constant of integration A_0 is related to the initial conditions by

$$A_0 = \sin^{-1}\left(\frac{\bar{\Omega}/|\Omega_0| - D_0}{\sqrt{D_0^2 - 1}}\right). \tag{36a}$$

Two convenient alternative forms of (36a) are

$$\sqrt{D_0^2 - 1}\,\sin A_0 = \frac{\bar{\Omega}}{|\Omega_0|} - D_0 = \frac{\bar{\Omega}}{2|\Omega_0|} - \frac{|\Omega_0|}{2\bar{\Omega}} - \frac{\delta_0^2}{8\bar{\Omega}|\Omega_0|}, \tag{36b}$$

$$\sqrt{D_0^2 - 1}\,\cos A_0 = \sqrt{2D_0\,\bar{\Omega}/|\Omega_0| - \bar{\Omega}^2/\Omega_0^2 - 1} = \frac{\delta_0}{2|\Omega_0|}. \tag{36c}$$

In view of (17) and (18) (and the fact that $|\Omega|/|\Omega_0| = \Omega/\Omega_0$), the divergence and angular velocity are given by

$$\delta = 2\bar{\Omega}\frac{\sqrt{D_0^2 - 1}\,\cos(2\bar{\Omega}t + A_0)}{D_0 + \sqrt{D_0^2 - 1}\,\sin(2\bar{\Omega}t + A_0)}, \tag{37}$$

$$\Omega = \bar{\Omega}\frac{\Omega_0/|\Omega_0|}{D_0 + \sqrt{D_0^2 - 1}\,\sin(2\bar{\Omega}t + A_0)}. \tag{38}$$

Equations (25) and (26) then yield the height and vertical velocity as

$$z = z_b(t) + \int_{z_b(0)}^{z_0} \frac{\bar{\Omega}/|\Omega_{0'}|}{D_{0'} + \sqrt{D_{0'}^2 - 1}\,\sin(2\bar{\Omega}t + A_{0'})}\,\mathrm{d}z_0', \tag{39}$$

$$H = \dot{z}_b(t) - \int_{z_b(0)}^{z_0} \frac{2\bar{\Omega}^2}{|\Omega_{0'}|} \frac{\sqrt{D_{0'}^2 - 1}\cos(2\bar{\Omega}t + A_{0'})}{\left[D_{0'} + \sqrt{D_{0'}^2 - 1}\sin(2\bar{\Omega}t + A_{0'})\right]^2}\, \mathrm{d}z'_0. \tag{40}$$

Although analytic evaluation of (39) and (40) will be possible only for the simplest profiles of divergence and angular velocity, some general observations can be made. According to (34), $D_0 = 1$ for any horizontal material surface on which the initial divergence is zero and the initial angular velocity has the same magnitude as the angular velocity in the primary vortex ($\Omega_0 = \pm\bar{\Omega}$). Such a surface remains non-divergent with $\Omega = \pm\bar{\Omega}$ for all time. However, (39) and (40) show that the height and vertical velocity of that surface depend on the initial vorticity and divergence distributions beneath it, and are generally oscillatory with a period $\pi/\bar{\Omega}$ equal to half the orbital period of fluid elements in the primary vortex. For a horizontal material surface on which $D_0 \neq 1$, the divergence and azimuthal velocities of that surface are also oscillatory with period $\pi/\bar{\Omega}$.

We also note that a forced oscillation of the lower boundary in the otherwise unbounded flow merely supports *en masse* lifting of the fluid. In contrast, forced motion by one horizontal boundary plate of a vortex confined at both the top and bottom is associated with a variable $C(t)$ and a more complicated motion (see §4).

The analytic solution leads to a simple breakdown result. If a horizontal material surface has zero initial angular velocity ($\Omega_0 = 0$), D_0 is infinite and (38) shows that the angular velocity of that surface remains zero for all time (a result that could also have been deduced from (16)). However, in view of (37), the divergence on that surface, $\delta = 2\bar{\Omega}\cos(2\bar{\Omega}t + A_0)/[1 + \sin(2\bar{\Omega}t + A_0)]$, becomes singular (denominator vanishes) within half the orbital period of the primary vortex ($t < \pi/\bar{\Omega}$). This means that any counter-rotating flow with a vertically continuous angular velocity distribution (that is, a counter-rotating flow containing a layer of non-rotating fluid) will blow up. If the initial divergence is zero, breakdown occurs at $t = \pi/(2\bar{\Omega})$.

Solution breakdown can be traced to the presence of the δ^2 term in (15), which in turn can be traced to the radial advection term $u\,\partial u/\partial r$ in the radial equation of motion (1). Equation (15) with $\Omega = 0$ is a Ricatti equation whose solution (with $C = 2\bar{\Omega}^2 > 0$) blows up in a finite time. If the δ^2 term were replaced by a linear term, the corresponding solution would not be singular. Physically, breakdown is due to the maintained impressing of the radial vortex pressure gradient force on a layer of non-rotating fluid. The radial acceleration sustained by this pressure gradient force eventually leads to singular radial convergence. In contrast, even a small initial value of angular velocity precludes a singularity in radial convergence. This is because the imbalance between the pressure gradient force and centrifugal force is reduced (and actually reverses) as the angular velocity spins up in the convergent inflow.

An example of solution breakdown was discussed in Shapiro & Markowski (1999) for a vertically discontinuous two-layer flow consisting of a primary vortex overlying a finite layer of non-rotating fluid. Breakdown of the solution of the von Kármán–Bödewadt similarity equations has also been described by Bodonyi & Stewartson (1977) and Bodonyi (1978) for viscous counter-rotating vortex flow. These two studies were concerned with the development of an unsteady viscous boundary layer on an infinite rotating disk in a rotating fluid. The boundary layer was created by suddenly reversing the sense of disk rotation, an action that induced counter-rotating flow adjacent to the disk and ensured the existence of a level of non-rotating fluid.

Breakdown of the Navier–Stokes equations also occurs in a class of Burgers-type vortex flows (Gibbon, Fokas & Doering 1999) in which the straining velocity field varies linearly with distance (as in the von Kármán–Bödewadt scaling) but with no such restriction on the rotational velocity component. The straining velocity in these Burgers-type flows mediates a balance between vorticity advection, stretching and diffusion, and results in a concentrated 'columnar' vortex that decays with radius in the limit of large radius. Solution breakdown is associated with a positive value of $\partial^2 p/\partial z^2$, which appears as a spatially uniform coefficient in a Riccati equation for the strain rate (a positive value of $\partial^2 p/\partial z^2$ would force an axial downflow, that is, a 'down-and-out' meridional circulation known to be inconsistent with a well-behaved Burgers model (Burgers 1948; Rott 1958)). In our present study, solution breakdown for $\Omega = 0$ on a material surface is associated with a positive spatially uniform coefficient $C\,(= 2\bar{\Omega}^2)$ in the Riccati equation for the vertical strain rate (equation (15) with $\Omega = 0$). The Riccati equations in the two investigations are formally identical, although our coefficient C is a measure of $\partial^2 p/\partial r^2$, rather than of $\partial^2 p/\partial z^2$.

3.2. *An example illustrating solution breakdown*

Consider an initially non-divergent boundary-layer vortex in which the initial angular velocity varies linearly with height from Ω_{s0} at the stationary lower boundary ($z_b(t) = 0$) to $\bar{\Omega}$ at the top of the boundary layer z_h (base of the primary vortex), that is

$$\left.\begin{aligned} \delta_0 &= 0, && 0 \leqslant z_0 < \infty, \\ \Omega_0 &= \bar{\Omega}[(1-\mu)z_0/z_h + \mu], && 0 \leqslant z_0 \leqslant z_h \\ &= \bar{\Omega}, && z_h < z_0 < \infty, \end{aligned}\right\} \tag{41}$$

where $\mu \equiv \Omega_{s0}/\bar{\Omega}$. We are particularly interested in the case where the fluid at the plate rotates in a sense opposite that of the fluid in the primary vortex ($\mu < 0$), a state that is qualitatively similar to the boundary layer that develops in a rotating viscous fluid shortly after a lower infinite disk has been forced to counter-rotate (Bodonyi & Stewartson 1977; Bodonyi 1978). In our case, a non-rotating material surface is present within the inviscid boundary layer at the initial height $z_0^* \equiv -\mu z_h/(1-\mu)$. Since there is no rotation on the z_0^* material surface, and the initial divergence is zero, the solution should blow up at $t = \pi/(2\bar{\Omega})$. As we will see, the nature of the breakdown in this counter-rotating flow is quite unusual.

Expanding the sine and cosine terms in (37) and (38) with addition formulas, and then applying (36*b*) and (36*c*) wherever possible leads to

$$\begin{aligned} \Omega &= \frac{2\bar{\Omega}\mu(1 - z_0/z_0^*)}{1 + \mu^2(1 - z_0/z_0^*)^2 + [1 - \mu^2(1 - z_0/z_0^*)^2]\cos 2\bar{\Omega}t}, && 0 \leqslant z_0 \leqslant z_h \\ &= \bar{\Omega}, && z_h < z_0 < \infty, \end{aligned} \tag{42}$$

$$\begin{aligned} \delta &= \frac{-2\bar{\Omega}[1 - \mu^2(1 - z_0/z_0^*)^2]\sin 2\bar{\Omega}t}{1 + \mu^2(1 - z_0/z_0^*)^2 + [1 - \mu^2(1 - z_0/z_0^*)^2]\cos 2\bar{\Omega}t}, && 0 \leqslant z_0 \leqslant z_h \\ &= 0, && z_h < z_0 < \infty, \end{aligned} \tag{43}$$

while (39) is evaluated as

$$\begin{aligned} z &= 2z_h \frac{\tan^{-1}[\mu(1 - z_0/z_0^*)\tan\bar{\Omega}t] - \tan^{-1}(\mu\tan\bar{\Omega}t)}{(1-\mu)\sin 2\bar{\Omega}t}, && 0 \leqslant z_0 \leqslant z_h \\ &= z_0 - z_h + T_1(t), && z_h < z_0 < \infty. \end{aligned} \tag{44}$$

The depth of the boundary layer vortex $T_1(t)$ is obtained by setting $z_0 = z_h$ in the first part of (44),

$$T_1(t) = 2z_h \frac{\bar{\Omega}t - \tan^{-1}(\mu \tan \bar{\Omega}t)}{(1-\mu)\sin 2\bar{\Omega}t}, \quad \text{for } t < \pi/(2\bar{\Omega}). \tag{45}$$

Equations (42)–(44) are a parametric form of the solution (with z_0 as a parameter). To obtain the Eulerian solution, solve (44) for z_0,

$$\begin{aligned} z_0 &= z_0^* - \frac{z_0^*}{\mu \tan \bar{\Omega}t} \tan\left[\frac{(1-\mu)z}{2z_h}\sin 2\bar{\Omega}t + \tan^{-1}(\mu \tan \bar{\Omega}t)\right], \quad z \leqslant T_1(t) \\ &= z + z_h - T_1(t), \qquad\qquad z > T_1(t), \end{aligned} \tag{46}$$

and use (46) to eliminate z_0 in favour of z and t in (42) and (43).

Inspection of (43) shows that $\delta(t, z_0) \to 0$ as $t \to \pi/(2\bar{\Omega})$ for all z_0, with the possible exception of the non-rotating material surface z_0^* where both the numerator and denominator of δ vanish. Further analysis shows that $\delta(t, z_0^*) = -2\bar{\Omega}\tan\bar{\Omega}t$, so $\delta(t, z_0^*) \to -\infty$ as $t \to \pi/(2\bar{\Omega})$. Thus, there is infinite convergence (radial inflow) along the non-rotating material surface z_0^*.

Equation (43) also shows that for all times prior to the onset of the singularity ($t < \pi/(2\bar{\Omega})$), δ has the same sign as $-2\bar{\Omega}[1 - \mu^2(1 - z_0/z_0^*)^2]$ (since the denominator in (43) is positive, and $\sin 2\bar{\Omega}t$ is positive for $t < \pi/(2\bar{\Omega})$). For μ in the 'weak' counter-rotation regime $-1 < \mu < 0$, it can readily be shown that $1 - \mu^2(1 - z_0/z_0^*)^2 > 0$, and so there is radial convergence ($\delta < 0$) and rising motion throughout the boundary layer. For μ in the 'strong' counter-rotation regime $\mu < -1$, we can identify a material surface $z_\delta \equiv z_0^*(1 - 1/|\mu|)$ on which the divergence is zero. The flow is divergent beneath this surface ($\delta > 0$ for $0 \leqslant z_0 < z_\delta$), and convergent above it ($\delta < 0$ for $z_\delta < z_0 \leqslant z_h$). The non-divergent surface z_δ is characterized by an angular velocity that is equal and opposite to the angular velocity in the primary vortex, $\Omega(t, z_\delta) = -\bar{\Omega}$. It is easily seen that such a material surface is consistent with (15) and (16).

Equation (44) shows that the denominator of z vanishes as $t \to \pi/(2\bar{\Omega})$, but the behavior of the numerator is obscured by the presence of two $\tan^{-1}$ functions. The arguments of these functions are each proportional to $\tan\bar{\Omega}t$, and thus become unbounded as $t \to \pi/(2\bar{\Omega})$. For $\mu < 0$ and t slightly less than $\pi/(2\bar{\Omega})$, the argument of the first $\tan^{-1}$ function is positive for $z_0 > z_0^*$ and negative for $z_0 < z_0^*$, while the argument of the second $\tan^{-1}$ function is negative. Since $\tan^{-1}\chi$ behaves like $\pi/2 - 1/\chi$ for large positive χ, and like $-\pi/2 - 1/\chi$ for large negative χ (Dwight 1961, equations (505.2) and (505.3)), we find that

$$\begin{aligned} \lim_{t \to \pi/(2\bar{\Omega})} z &= \frac{z_0}{\mu^2(1 - z_0/z_0^*)}, \quad 0 \leqslant z_0 < z_0^* \\ &= \infty, \qquad\qquad z_0^* \leqslant z_0 < \infty. \end{aligned} \tag{47}$$

The extension of the inequality in (47) to ∞ follows from the fact that the boundary-layer thickness T_1 becomes singular.

Taking the time derivative of (45), and examining the result for t near $\pi/(2\bar{\Omega})$, we see that the normal velocity of the top of the boundary layer just prior to blow-up behaves as

$$\frac{dT_1}{dt} \approx \frac{\pi z_h \bar{\Omega}}{(1-\mu)\cos^2 \bar{\Omega}t} \approx \frac{\pi z_h \bar{\Omega}}{1-\mu}(\pi/2 - \bar{\Omega}t)^{-2}, \quad \text{for } t \text{ near } \pi/(2\bar{\Omega}). \tag{48}$$

Equation (42) describes a discontinuous blow-up of the angular velocity Ω near z_0^*:

$$\begin{aligned}\lim_{t\to\pi/(2\bar{\Omega})} \Omega &= \frac{\bar{\Omega}}{\mu(1-z_0/z_0^*)} < 0, \quad 0 \leqslant z_0 < z_0^* \\ &= 0, \qquad\qquad\qquad\quad z_0 = z_0^* \\ &= \frac{\bar{\Omega}}{\mu(1-z_0/z_0^*)} > 0, \quad z_0^* < z_0 \leqslant z_h \\ &= \bar{\Omega}, \qquad\qquad\qquad\quad z_h < z_0 < \infty. \end{aligned} \tag{49}$$

This angular velocity distribution has a simple physical interpretation. Since Ω is originally zero on the z_0^* material surface, it must remain zero. But the initially small values of Ω immediately adjacent to this surface spin up to infinite values in a convergent inflow that is singular on the z_0^* surface. Since Ω was originally positive just above the non-rotating surface, it becomes infinite through positive values. Similarly, immediately beneath the non-rotating surface, Ω becomes infinite through negative values.

Our breakdown results tend to support Bodonyi & Stewartson's (1977) hypothesis that breakdown of viscous counter-rotating flow is largely due to inviscid processes. The time of the inviscid breakdown, $t = \pi/(2\bar{\Omega}) \approx 1.571/\bar{\Omega}$, is in good agreement with the viscous breakdown time, $t \approx 2.365/\bar{\Omega}$. The violence of the inviscid breakdown is also similar to that reported in Bodonyi & Stewartson (1977), where all the velocity components become infinite. Of particular note, the normal velocity at the top of the boundary layer just prior to breakdown in both viscous and inviscid flows is proportional to $(t_E - \bar{\Omega}t)^{-2}$, where t_E is the breakdown time (compare (48) to (2.5) of Bodonyi & Stewartson).

Although our focus in this section has been on the solution breakdown attending counter-rotating flow ($\mu < 0$), we note that (42)–(44) is well behaved if attention is restricted to strictly positive angular velocity profiles ($\mu > 0$). In that case z_0^* does not lie within the boundary layer, and the denominators of δ and Ω never vanish. Although the denominator of z still vanishes, the numerator of z vanishes in such a manner that z remains finite. This well-behaved solution is similar to the example considered in the next section.

3.3. *An example of a well-behaved solution*

Next consider an initially non-divergent vortex in which the initial angular velocity varies exponentially from the lower boundary up to the base of the primary vortex,

$$\left.\begin{aligned}\delta_0 &= 0, & 0 \leqslant z_0 < \infty, \\ \Omega_0 &= \bar{\Omega}\exp(z_0/z_h - 1), & 0 \leqslant z_0 \leqslant z_h \\ &= \bar{\Omega}, & z_h < z_0 < \infty. \end{aligned}\right\} \tag{50}$$

Evaluation of (37) and (38) yields the angular velocity and divergence as

$$\begin{aligned}\Omega &= \bar{\Omega}\frac{1}{\cosh(1-z_0/z_h) + \sinh(1-z_0/z_h)\cos 2\bar{\Omega}t}, \quad 0 \leqslant z_0 \leqslant z_h \\ &= \bar{\Omega}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad z_h < z_0 < \infty, \end{aligned} \tag{51}$$

$$\delta = -2\bar{\Omega}\frac{\sin 2\bar{\Omega}t}{\coth(1 - z_0/z_h) + \cos 2\bar{\Omega}t}, \quad 0 \leqslant z_0 \leqslant z_h$$
$$= 0, \qquad z_h < z_0 < \infty, \tag{52}$$

and evaluation of (39) yields

$$z = \frac{z_h}{2}\sec^2\bar{\Omega}t \ln\left[\frac{\tan^2\bar{\Omega}t + \mathrm{e}^2}{\tan^2\bar{\Omega}t + \exp\left[2(1 - z_0/z_h)\right]}\right], \quad 0 \leqslant z_0 \leqslant z_h$$
$$= z_0 - z_h + T_1(t), \qquad z_h < z_0 < \infty, \tag{53}$$

where the depth of the boundary layer vortex is $T_1(t) = \frac{1}{2}z_h \sec^2\bar{\Omega}t \ln[1 + (\mathrm{e}^2 - 1)\cos^2\bar{\Omega}t]$. It can readily be verified that these expressions are always bounded.

To obtain the Eulerian solution, invert (53) to obtain an explicit formula for z_0,

$$z_0 = z_h - \tfrac{1}{2}z_h \ln[(\tan^2\bar{\Omega}t + \mathrm{e}^2)\exp(-2\cos^2\bar{\Omega}t\, z/z_h) - \tan^2\bar{\Omega}t], \quad z \leqslant T_1(t)$$
$$= z + z_h - T_1(t), \qquad T_1(t) < z, \tag{54}$$

and substitute this form in (51) and (52). This solution is displayed in figure 1. The flow consists of a centrifugally driven 'sloshing' motion in the vertical, with compensating low-level radial inflow and outflow. As the convergence in the lower vortex oscillates between peak positive and negative values, the stretching process forces the vorticity to alternately amplify and attenuate.

4. Solutions for variable $C(t)$: vertically confined vortex forced by top plate

4.1. *Uniform angular velocity and divergence*

Now consider the special case of a vortex of initially uniform angular velocity and divergence confined between two impermeable horizontal plates. The lower plate is stationary and the top plate undergoes an arbitrary vertical acceleration. The initial divergence must be chosen to be consistent with this top plate motion. In view of the meridional vorticity conservation equation (31), the initial uniform conditions for δ and Ω ensure that δ and Ω remain uniform for all time.

With the plate motion specified by

$$z_b(t) = 0, \quad z_t(t) = f(t), \tag{55}$$

and the initial conditions given by

$$\Omega(0, z) = \bar{\Omega}, \quad \delta(0, z) = -\dot{f}(0)/f(0), \tag{56}$$

the vertical velocity, divergence and angular velocity fields are readily obtained as

$$H = z\dot{f}(t)/f(t), \quad \delta = -\dot{f}(t)/f(t), \quad \Omega = \bar{\Omega}f(t)/f(0). \tag{57}$$

For completeness we note that $C(t)$ can be evaluated as

$$C(t) = \ddot{f}/f + 2\bar{\Omega}^2 f^2/f^2(0) - \tfrac{3}{2}\dot{f}^2/f^2. \tag{58}$$

4.2. *Two-layer vortex*

Now consider a two-layer vortex in which the initial angular velocity and divergence are uniform in each layer and the bounding plates satisfy (55). In view of (31), δ and Ω remain uniform in each layer for all time. We denote the angular velocities in the

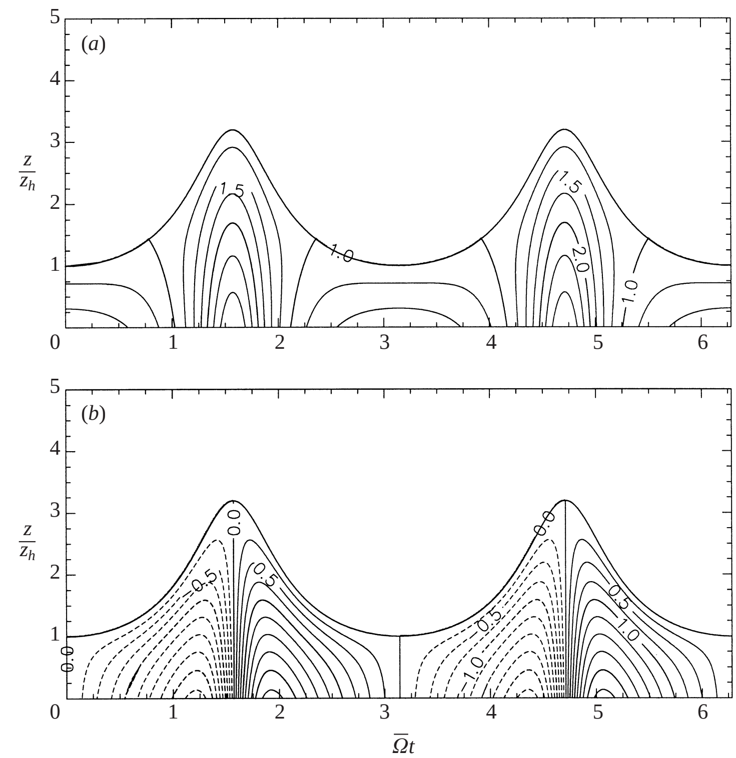

FIGURE 1. Contour plots of (a) angular velocity $\Omega/\bar{\Omega}$ and (b) divergence $\delta/\bar{\Omega}$ for a vortex with initial conditions specified in (50). Angular velocity increases with height from a stationary lower plate up to the base $z = T_1(t)$ of a uniformly rotating primary vortex. All quantities are non-dimensional. Contour increment in (a) and (b) is 0.25. Negative contours are dashed.

lower and upper layers by $\Omega_1(t)$ and $\Omega_2(t)$, respectively. The layer divergences $\delta_1(t)$ and $\delta_2(t)$ are defined analogously.

Following the approach of Shapiro & Markowski (1999) (for stationary plates) we obtain

$$\Omega_1(t) = \frac{\Omega_1(0)}{T_1(0)} T_1(t), \quad \Omega_2(t) = \frac{\Omega_2(0)}{f(0) - T_1(0)} (f(t) - T_1(t)), \tag{59a,b}$$

$$\delta_1 = -\frac{1}{T_1} \frac{\mathrm{d}T_1}{\mathrm{d}t}, \quad \delta_2 = \frac{1}{f - T_1} \left(\frac{\mathrm{d}T_1}{\mathrm{d}t} - \dot{f} \right), \tag{60a,b}$$

where T_1 is the height of the interface between the two layers. Applying these expressions in (15) for each layer, and then subtracting one equation from the other yields a second-order nonlinear ordinary differential equation

$$\frac{\mathrm{d}^2 T_1}{\mathrm{d}t^2} + \frac{3}{2} \left(\frac{1}{f - T_1} - \frac{1}{T_1} \right) \left(\frac{\mathrm{d}T_1}{\mathrm{d}t} \right)^2 - 2 \left(\frac{\Omega_2(0)}{f(0) - T_1(0)} \right)^2 \frac{T_1}{f} (f - T_1)^3 + 2 \left(\frac{\Omega_1(0)}{T_1(0)} \right)^2 \left(1 - \frac{T_1}{f} \right) T_1^3 + \frac{3}{2} \frac{T_1}{f - T_1} \frac{\dot{f}}{f} \left(\dot{f} - 2 \frac{\mathrm{d}T_1}{\mathrm{d}t} \right) - \frac{T_1}{f} \ddot{f} = 0. \tag{61}$$

The initial value problem consists of solving (61) with specified $f(t)$, and initial data $T_1(0)$, $\mathrm{d}T_1/\mathrm{d}t(0), \Omega_1(0)$ and $\Omega_2(0)$. The initial value $\mathrm{d}T_1/\mathrm{d}t(0)$ can be specified in terms of the divergences in either layer via (60a) or (60b). However, just as the initial

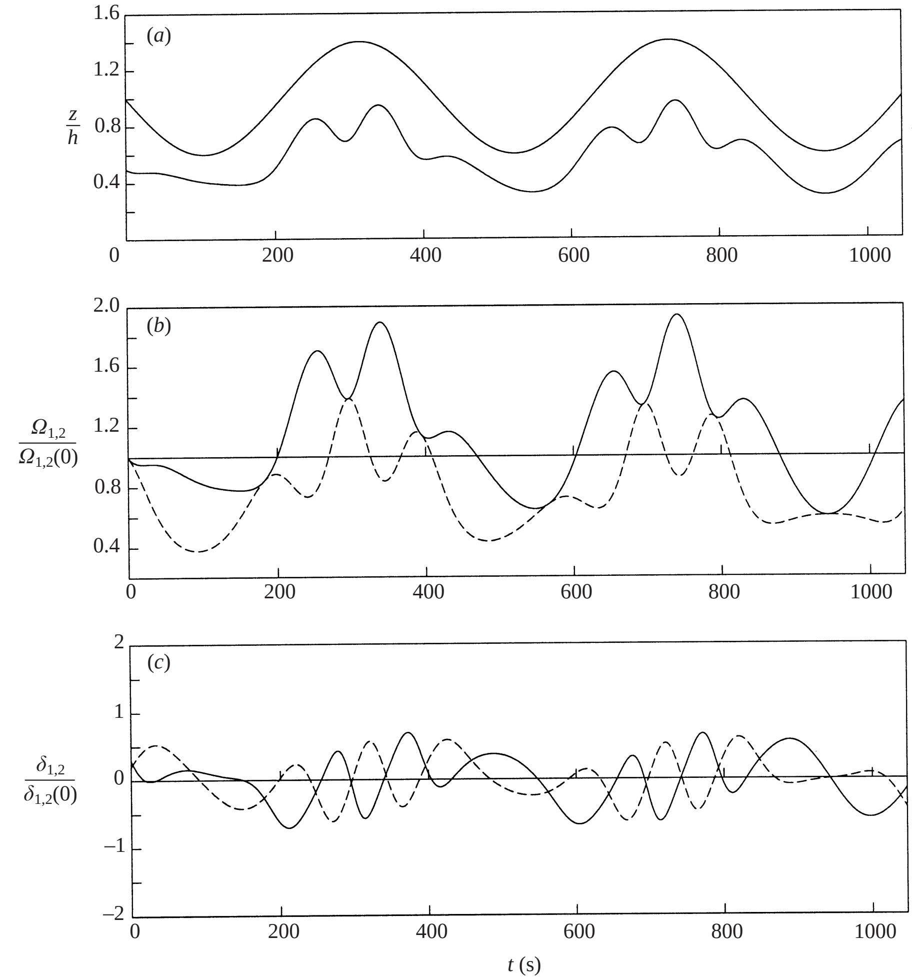

FIGURE 2. Evolution of a two-layer vortex forced by a vertical oscillation of the top plate with sub-inertial frequency $\omega = 0.015\,\mathrm{s}^{-1}$: (*a*) interface height $T_1(t)/h$ (lower curve) and top plate height $f(t)/h$ (top curve); (*b*) lower-layer angular velocity $\Omega_1(t)/\Omega_1(0)$ (solid curve) and upper-layer angular velocity $\Omega_2(t)/\Omega_2(0)$ (dashed curve); (*c*) lower-layer divergence $\delta_1(t)/\delta_1(0)$ (solid curve) and upper-layer divergence $\delta_2(t)/\delta_2(0)$ (dashed curve).

divergence in the single-layer example could not be chosen arbitrarily, $\delta_1(0)$ and $\delta_2(0)$ in the two-layer vortex cannot be chosen independently of each other. Elimination of $\mathrm{d}T_1/\mathrm{d}t$ from (60*a*) and (60*b*) yields

$$\delta_2 = -\frac{T_1}{f - T_1}(\delta_1 + \dot{f}/T_1). \tag{62}$$

The initial conditions must be consistent with this condition.

In the case where the top plate is stationary (so $f(t) = h$, a constant), (61) is autonomous and can be solved analytically (Shapiro & Markowski 1999). This unforced (inertial) solution is found to be periodic with natural frequency

$$\omega_{\mathrm{nat}} = 2\left(\frac{1 - T_1(0)/h}{|\Omega_2(0)|} + \frac{T_1(0)/h}{|\Omega_1(0)|}\right)^{-1} = 2h\left(\int_0^h \frac{1}{|\Omega(z,0)|}\,\mathrm{d}z\right)^{-1}. \tag{63}$$

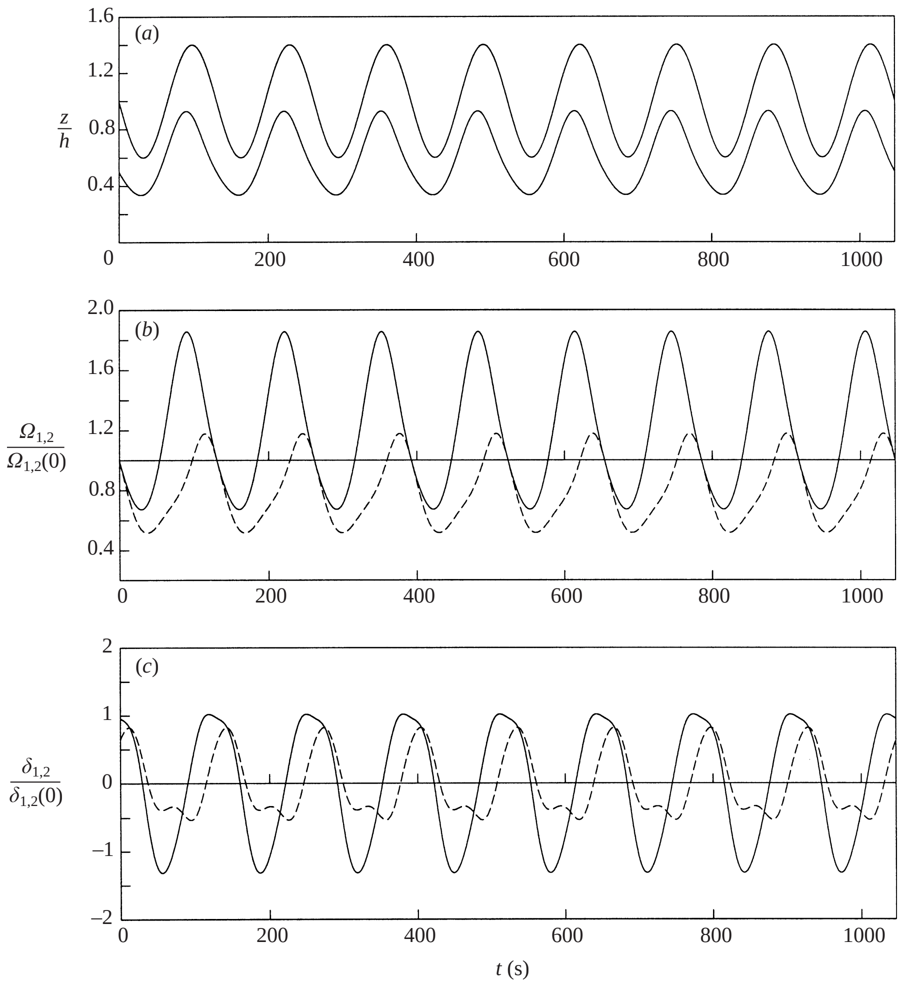

FIGURE 3. As figure 2 but for the natural frequency $\omega = \omega_{\text{nat}} = 0.048\,\text{s}^{-1}$.

Equation (24) can be used to show that (63) remains valid if we replace $\Omega(z, 0)$ by $\Omega(z, t)$, that is, ω_{nat} is independent of the time at which the angular velocity is evaluated. Since the orbital period of a fluid parcel (layer) with angular velocity Ω is $2\pi/\Omega$, the natural period of oscillation, $2\pi/\omega_{\text{nat}}$, is half the mean orbital period of the vortex. For a non-stationary plate we will appeal to numerical methods of solution and present results for a length of time of a fixed number of the corresponding natural periods of oscillation.

Now consider the evolution of a two-layer vortex forced by a top plate oscillation of the form $f(t) = h(1 - \alpha \sin \omega t)$. With attention restricted to initial layer thicknesses and divergences that are the same in each layer, we find that $T_1(0) = h/2$, $\delta_1(0) = \delta_2(0) = \alpha\omega$, and $\text{d}T_1/\text{d}t(0) = -\alpha\omega h/2$. We fix the oscillation amplitude at $\alpha = 0.4$ and set the initial angular velocities in the lower and upper layers to $\Omega_1(0) = 0.02\,\text{s}^{-1}$ and $\Omega_2(0) = 0.03\,\text{s}^{-1}$, respectively. The corresponding natural (inertial) frequency is obtained from (63) as $\omega_{\text{nat}} = 0.048\,\text{s}^{-1}$ (this is the frequency that would be exhibited by the two-layer vortex if the top plate were stationary). The solution of this forced problem is obtained by breaking (61) into two first-order equations that are then

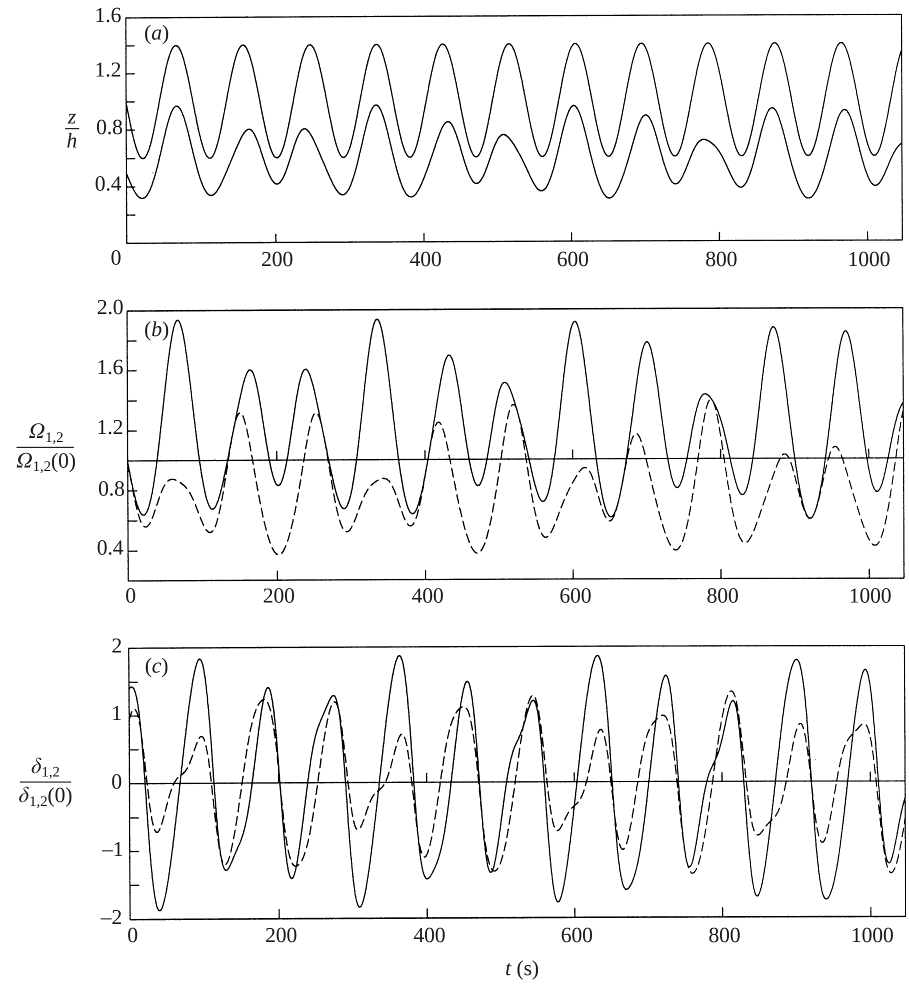

FIGURE 4. As figure 2 but for a super-inertial frequency $\omega = 0.07\,\mathrm{s}^{-1}$.

integrated with a standard fourth-order Runge–Kutta method (Press *et al.* 1992, § 16.1). The integration proceeds for 1047.2 s (eight periods of the corresponding natural oscillation) with a time-step size of $\Delta t = 0.5236$ s. We present results for a sub-inertial frequency of $\omega = 0.015\,\mathrm{s}^{-1}$ (figure 2), the natural frequency $\omega = \omega_{nat} = 0.048\,\mathrm{s}^{-1}$ (figure 3), and a super-inertial frequency of $\omega = 0.07\,\mathrm{s}^{-1}$ (figure 4). For comparison, the corresponding stationary plate case is displayed in figure 5. Although the solutions are generally oscillatory with several frequencies present, no solutions were found with irregular or broadband structures characteristic of chaos. Forcing at the natural frequency here and in experiments with other parameter choices (not shown) failed to elicit resonance.

5. Summary remarks

We consider the class of radially unbounded solid-body-type vortices in which the angular velocity varies along the rotation axis. The flows are assumed to be inviscid incompressible and axisymmetric. The von Kármán–Bödewadt velocity scalings apply

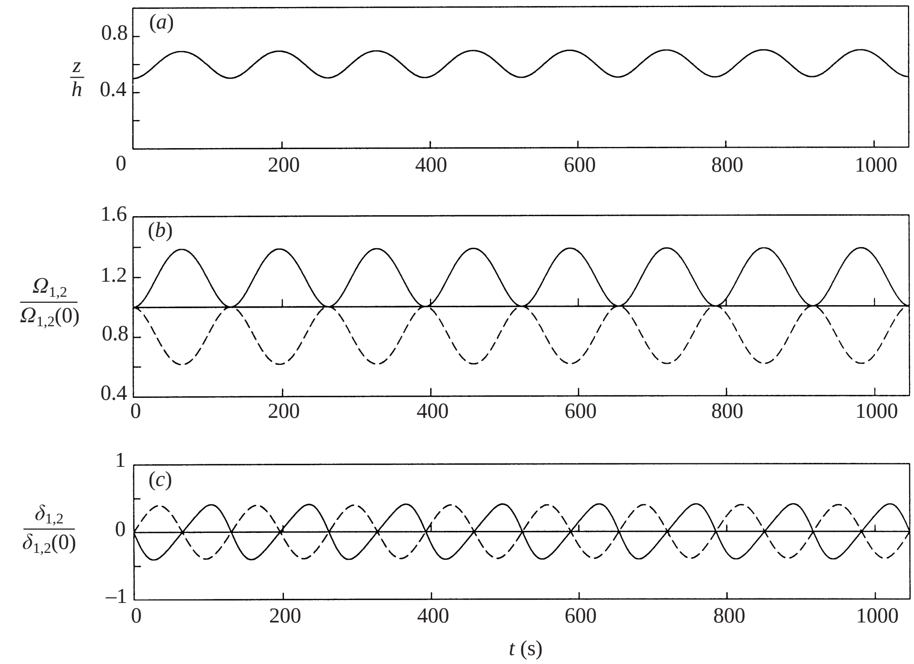

FIGURE 5. As figure 2 but for the unforced vortex. In (a) the top plate is stationary at $z/h = 1$, and the interface height $T_1(t)/h$ is oscillatory. Plotting scales are the same as in figures 2–4.

to this flow class and lead to exact solutions of the Euler equations for several scenarios. In particular, a general closed-form solution is obtained for a vertically infinite vortex of uniform angular velocity overlying a bounded vortex (bounded from below by a horizontal plate) with a variable angular velocity distribution. Although such a flow is generally oscillatory, the solution blows up in a finite time (less than half the orbital period of the primary vortex) if the fluid at any level is not rotating. The example considered in § 3.2 tends to support Bodonyi & Stewartson's (1977) hypothesis that breakdown of viscous Bödewadt-type counter-rotating vortex flows over a rotating disk may be largely inviscid in nature.

Provision is also made for boundary forcing from vertically oscillating horizontal plates. The solution for a vertically confined single-layer (uniform) vortex forced by the motion of the top plate is extremely simple and is dominated by the form of the boundary forcing (vertical vorticity is proportional to boundary displacement $f(t)$). Numerical solutions are presented for a vertically confined two-layer vortex forced by a top plate in simple harmonic motion. These latter solutions do not appear to be chaotic.

The stability of these solutions has not been explored. If general perturbations are considered (perturbations unconstrained by the similarity relations), shear instabilities would be anticipated to develop in some cases; however, the question of stability is beyond the scope of the present investigation.

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